PHYS1110D – Engineering Physics: Mechanics and Thermodynamics

Tutorial Problems for Week 3: Differentiation and Integration

**Formulae You Should Memorize:**

**Problem 1 – Linear Motion with Changing Acceleration**

1. A body is moving on the axis. At time , it is at the origin and is moving at velocity 10 m/s2 towards the direction. Find the position for if its acceleration is given by:

a) ; b) (and in m/s2 and seconds respectively.)

1. A body is moving on the axis. Find the position and acceleration for if the velocity of the body is given by

a) ; b) (and in m/s and seconds respectively)

**Solution:**

Solving this problem is in principle simple:

However, we have to remind you an important detail: whenever you are going to integrate, don’t forget the *initial condition*! They correspond to the *integration constants* when calculating indefinite integrals.

* 1. (*Again, don’t forget !*)

*Remark*: You can try applying the rule for differentiating the product of two functions

when finding the acceleration. Check if the results agree with each other.

**Problem 2 – The Chain Rule of Differentiation**

**(You can apply this rule when solving this week’s assignment)**

For a “reasonable” function , if is again a function of another variable (i.e. ), and we want to calculate the derivative of with respect to , the **chain rule** states that:

To physicists’ level of rigor, the rule is almost self-evident (you must be tempted to cross out the two ’s). For example, if , then

As one of its simple applications, assume the position of a particle in the -plane is given by

Now, provided that *(you’d better memorize these as well)*

Find the velocity and the acceleration as a function of time. Do you recover something familiar?

**Solution:**

Because the basis vectors in the -coordinate system does not change with time, we can leave them alone, and differentiate/integrate the vector components directly. Therefore

How to deal with ? Well, using the chain rule, we get

Similarly

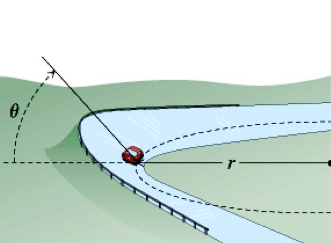
Therefore

Does it look familiar to you? We see that is perpendicular to , as it should be. And the magnitude of velocity is

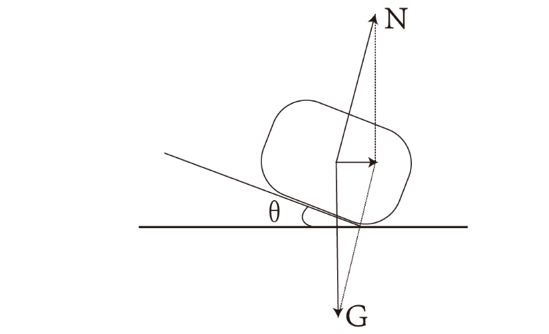
Familiar! Finding the acceleration is now easy:

This is in the *opposite* direction of : it is exactly the *centripetal acceleration*! Its magnitude is

which is also a result you already know.

**Problem 3 – Circular Motion on Banked Road**

Engineers often make use of the centripetal acceleration when designing road turnings. Suppose a turning is an arc of radius (see the figure). Cars are expected to pass this turning with speed . In order to reduce the friction along the radial direction between the car tires and the road to zero, at what angle should the road be banked?

**Solution:**

The free-body diagram of the car when passing the turning is shown on the left. The centripetal force is entirely provided by the radial (horizontal) component of the normal force exerted by the road on the car. Meanwhile, the normal force must balance the gravity in the vertical direction, since the car is not moving vertically. Now we can write down

**Problem 4\*[[1]](#footnote-1) – The Principle of Least Action**

*Before diving into the final problem, we have to teach you some multi-variable calculus. You already know that for a one-variable function , its extremum point must satisfy the condition:*

*This can be easily generalized to the multi-variable case: for a function of N-variables , its extremum point must satisfy a set of conditions simultaneously:*

*The symbol means that we are taking the derivative with respect to , while treating other variables as constants. As an elementary example,*

A guy on the axis has potential energy . He decides to start moving at time from , and reach at time . Furthermore, he even makes a detailed plan on how to get there, which can be described as follows:

He divides the time interval into parts. The th part () corresponding to , with . In each time interval, he moves with *constant velocity* and reaches at the end of each interval (of course, ). Then he takes the limit of , so his move will be smooth (without sudden change of velocity at the end of each interval). [[2]](#footnote-2) How does he choose the values of ? Well, he designed a special integral called the **action**[[3]](#footnote-3):

Since he moves in each interval at constant velocity, we can divide the action into parts, and write

Where we have used to approximate the potential energy in the region . The action is thus a function of variables . His criterion for the “best” way of moving is called **the Principle of Least Action**:

*The best way to move corresponds to the least action*

Now, please translate this statement into math equations using what you have just learned about the extremum of multi-variable functions. As , what do you get?

**Solution:**

The Principle of Least Action is just a set of extremum conditions

We first calculate the action integral: by definition (we omit the “lim” symbol for a while)

Therefore (*Be careful about where appears in the sum. You may try some concrete examples, e.g.* )

Interesting things happen as we take the limit :

This is the *velocity* of the guy at time ! Then (we change to )

The left-hand side contains the expression

It is the *acceleration* at time ! [[4]](#footnote-4)

A final problem: the acceleration is evaluated at , but is at . However, as , the difference between and can be neglected. Now we can finally write down the equation that governs how the guy moves:

This is the famous *Newton’s Second Law*[[5]](#footnote-5). Does it come out as a surprise to you?

*Remarks:*

* The integrand in the action is called the **Lagrangian**. In our example
* We emphasize that the action is compared among paths with *given starting and ending point (including position and time)*. We cannot compare one path to another with different starting point or ending point. Therefore, to some extent, we can say that the Principle of Least Action makes Newton’s Law *meaningful*: the particle seems to “*know*” *where* it wants to go and *when* it plans get there, then it just “*chooses*” one path with the least action.
* The expression of the Lagrangian seems to come out of blue. In fact (especially for a free particle), the Lagrangian can be *derived* by some clever assumptions (consistent with experiments) about the properties of the spacetime we are living in. For example, a *free* particle feels that there is *neither special direction nor special position in space*, and it *does not know when did the time begin*. These statements[[6]](#footnote-6) are enough to lead us to . But how to do this is too far away from what an engineering student are expected to learn.

In addition, if the space is no longer the same everywhere (described by some *potential energy* function , then the Lagrangian is modified to the form you have already met.

* The Principle of Least Action is in fact a *more fundamental* law of nature than Newton’s Laws, which can lead us to relativistic mechanics and quantum mechanics.

*Recommended Reference:*

If you want to see the standard way to learn the Principle of Least Action, we recommend these books for you to read *after learning multi-variable calculus*:

* [Chapter 19 in the *Feynman Lectures on Physics* (Vol. 2)](http://www.feynmanlectures.caltech.edu/II_19.html)

By the way, the Principle of Least Action inspired Richard Feynman to develop the *path integral* formulation of quantum mechanics, which is a *very crazy* description of nature! How crazy? Roughly speaking, it says that, instead of solving differential equations like we humans do, nature *try all paths* (hence the name “*path* integral”) to determine which one has the least action!

* Chapter 1 in *Mechanics (Course of Theoretical Physics, Vol. 1)* by Landau and Lifshitz

This is one of the famous textbooks for physics students to learn the Principle of Least Action. It will tell you how to *derive* the Lagrangian of a free particle.

* Chapter 22 in *Mathematical Methods for Physics and Engineering*.

This is suitable for those who want to learn about the math behind the Principle (called the *calculus of variations*).

1. Problems with an asterisk may be difficult. Such problems are not necessary when studying for the exams. [↑](#footnote-ref-1)
2. Similar to what we did when learning integrals, huh? [↑](#footnote-ref-2)
3. We suspect that the name “action” is chosen to emphasize people’s tendency towards laziness. [↑](#footnote-ref-3)
4. There may be some steps in the solution that requires a good understanding on the meaning of taking limit. If you are confused, we advise you to seek help from your math teacher and teaching assistants. [↑](#footnote-ref-4)
5. The force related to the potential energy is , as you will learn in Week 5 of this term. [↑](#footnote-ref-5)
6. Or using some physics jargons: *isotropy of the space, translational symmetry of the space, and translational symmetry of the time.* [↑](#footnote-ref-6)